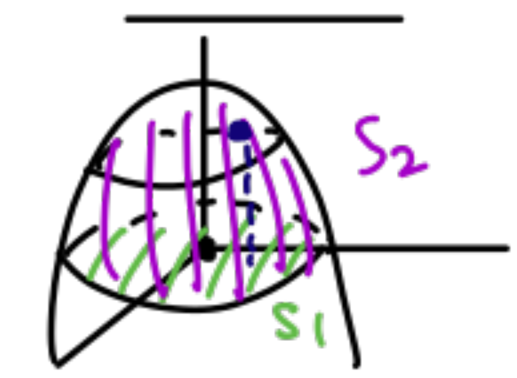
Ex: Compute the Flux of $\vec{F} = \langle y, x, z \rangle$ across the boundary of the solid enclosed by paraboloid $z = 1 - x^2 - y^2$ and plane $z=0$

Sol: $\iint_S \vec{F}\, d\vec{S} = \iint_D \vec{F} \cdot (\vec{r}_u \times \vec{r}_v)\, dA$

↑ Domain of parameterization $\vec{r}(u,v)$ for surface S.



parameterize S_1: $\vec{r}(u,v) = \langle u\cos(v), u\sin(v), 0 \rangle$ on $D_1 = [0,1] \times [0, 2\pi]$

parameterize S_2: $\vec{s}(u,v) = \langle u\cos(v), u\sin(v), 1-u^2 \rangle$ (r,θ) on $D_2 = [0,1] \times [0, 2\pi]$

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_{D_1} \vec{F} \overset{-}{\cdot} (\vec{r}_u \times \vec{r}_v)\, dA + \iint_{D_2} \vec{F} \overset{+}{\cdot} (\vec{s}_u \times \vec{s}_v)\, dA$$

consider the orientation:

$\vec{r}_u = \langle \cos(v), \sin(v), 0 \rangle$ $\quad \vec{r}_v = \langle -u\sin(v), u\cos(v), 0 \rangle$

$$\therefore \vec{r}_u \times \vec{r}_v = \det \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos(v) & \sin(v) & 0 \\ -u\sin(v) & u\cos(v) & 0 \end{bmatrix}$$

$= \langle 0, 0, u \rangle \longrightarrow$ orientation is inward here, so negate.

$\vec{s}_u = \langle \cos(v), \sin(v), -2u \rangle$

$\vec{s}_v = \langle -u\sin(v), u\cos(v), 0 \rangle$

$$\therefore \vec{s}_u \times \vec{s}_v = \det \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos(v) & \sin(v) & -2u \\ -u\sin(v) & u\cos(v) & 0 \end{bmatrix}$$

$= u\langle 2u\cos(\theta),\ 2u\sin(\theta), 1\rangle$

looking at $u=\frac{1}{2}$ again, this oriented outward.

$\therefore \vec{F}$ on S_1 is given by $\vec{F}(\vec{r}(u,\theta))$

$= \langle u\sin(\theta), u\cos(\theta), 0\rangle$

$\therefore$ on S_1, $\vec{F}(\vec{r}(u,\theta))\cdot(\vec{r}_u \times \vec{r}_\theta) = 0$

<u>And</u>: $\vec{F}$ on S_2 is given by:

$\vec{F}(\vec{s}(u,\theta)) = \langle u\sin(\theta), u\cos(\theta), 1-u^2\rangle$

$\therefore \vec{F}(\vec{s}(u,\theta))\cdot(\vec{s}_u \times \vec{s}_\theta) = u(2u^2\sin(\theta)\cos(\theta)$
$+ 2u^2\sin(\theta)\cos(\theta)+1-u^2)$

$\iint_S \vec{F}\cdot d\vec{s} = \iint_{D_1} \vec{F}\cdot(\vec{r}_u \times \vec{r}_\theta)\, dA + \iint_{D_2} \vec{F}\cdot(\vec{s}_u \times \vec{s}_\theta)\, dA$

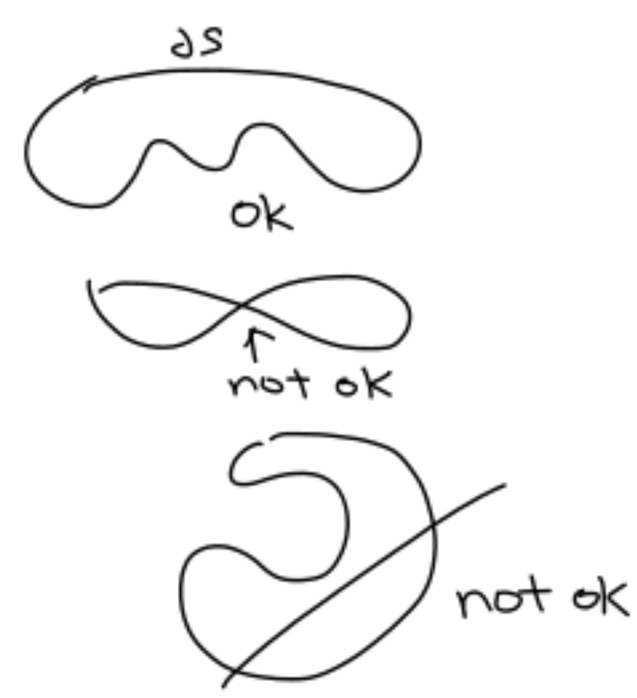
$= \iint_D 0\ dA + \iint_D u(4u^2\sin(\theta)\cos(\theta)+1-u^2)\, dA$

$= \frac{\pi}{2}$

IDEA: Generalize Green's Theorem to surfaces which are not flat...

Prop (Stokes's Theorem):

Suppose S is a piecewise smooth surface with piecewise-smooth boundary curve. which is closed and has only one component. If a vector field with continuous partial derivatives on S, then $\iint_S \operatorname{curl}(\vec{F}) \cdot d\vec{S} = \int_{\partial S} \vec{F}\, d\vec{r}$



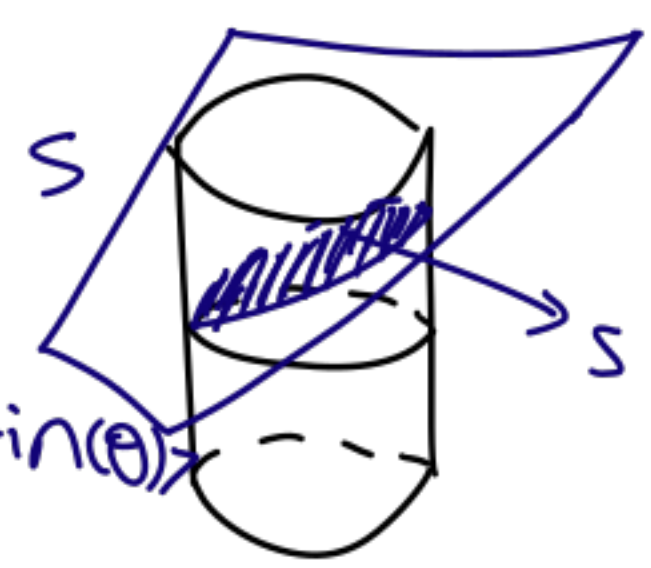
Ex: Compute $\int_C \vec{F} \cdot d\vec{r}$ for $\vec{F} = \langle -y^2, x, z^2 \rangle$ and C the curve of intersections of plane $y+z=2$ and cylinder $x^2+y^2=1$

Sol: We need $C = \partial S$ for some surface S

A good choice:

$\vec{S}(r,\theta) = \langle r\cos(\theta), r\sin(\theta), 2 - r\sin(\theta) \rangle$

on $(r,\theta) \in [0,1] \times [0, 2\pi]$



By Stokes's Theorem:

$$\int_C \vec{F} \cdot d\vec{r} = \int_{\partial S} \vec{F} \cdot d\vec{r} = \iint_C \operatorname{curl}(\vec{F}) \cdot d\vec{S}$$

$$= \iint_D \operatorname{curl}(\vec{F})(\vec{S}(r,\theta)) \cdot (\vec{S}_u \times \vec{S}_\theta)\, dA$$

$$\operatorname{curl}(\vec{F}) = \nabla \times \vec{F} = \det \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -y^2 & x & z^2 \end{vmatrix} = \langle 0, 0, 1+2y \rangle$$

$\text{curl}(\vec{F})(S(r,\theta)) = \langle 0, 0, 1+2r\sin(\theta) \rangle$

$\vec{S}_r = \langle \cos(\theta), \sin(\theta), -\sin(\theta) \rangle$

$\vec{S}_\theta = \langle -r\sin(\theta), r\cos(\theta), -r\cos(\theta) \rangle$

$$\therefore \vec{S}_r \times \vec{S}_\theta = \det \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos(\theta) & \sin(\theta) & -\sin(\theta) \\ -r\sin(\theta) & r\cos(\theta) & -r\cos(\theta) \end{bmatrix}$$

$$= r\langle 0, 1, 1 \rangle$$

has correct orientation for counterclockwise from above.

$$\therefore \int_C \vec{F} \cdot d\vec{r} = \iint_D \langle 0, 0, 1+2r\sin(\theta) \rangle \cdot r\langle 0, 1, 1 \rangle \, dA$$

$$= \int_{r=0}^{1} \int_{\theta=0}^{2\pi} r(1+2r\sin(\theta)) \, d\theta \, dr$$

$$= \pi$$

<u>Exercise</u>: Directly compute the line integrals...